

Connections between discriminants and the root distribution of polynomials with rational generating function

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Abstract

Let $H_m(z)$ be a sequence of polynomials whose generating function $\sum_{m=0}^{\infty} H_m(z)t^m$ is the reciprocal of a bivariate polynomial $D(t, z)$. We show that in the three cases $D(t, z) = 1 + B(z)t + A(z)t^2$, $D(t, z) = 1 + B(z)t + A(z)t^3$ and $D(t, z) = 1 + B(z)t + A(z)t^4$, where $A(z)$ and $B(z)$ are any polynomials in z with complex coefficients, the roots of $H_m(z)$ lie on a portion of a real algebraic curve whose equation is explicitly given. The proofs involve the q -analogue of the discriminant, a concept introduced by Mourad Ismail.

1 Introduction

In this paper we study the root distribution of a sequence of polynomials satisfying one of the following three-term recurrences:

$$\begin{aligned} H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-2}(z) &= 0, \\ H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-3}(z) &= 0, \\ H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-4}(z) &= 0, \end{aligned}$$

with certain initial conditions and $A(z), B(z)$ polynomials in z with complex coefficients. For the study of the root distribution of other sequences of polynomials that satisfy three-term recurrences, see [8] and [10]. In particular, we choose the initial conditions so that the generating function is

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{D(t, z)}$$

where $D(t, z) = 1 + B(z)t + A(z)t^2$, $D(t, z) = 1 + B(z)t + A(z)t^3$, or $D(t, z) = 1 + B(z)t + A(z)t^4$. We notice that the root distribution of $H_m(z)$ will be the same if we replace 1 in the numerator by any monomial $N(t, z)$. If $N(t, z)$ is not a monomial, the root distribution will be different. The quadratic case $D(t, z) = 1 + B(z)t + A(z)t^2$ is not difficult and it is also mentioned in [13]. We

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present this case in Section 2 because it gives some directions to our main cases, the cubic and quartic denominators $D(t, z)$, in Sections 3 and 4.

Our approach uses the concept of q -analogue of the discriminant (q -discriminant) introduced by Ismail [12]. The q -discriminant of a polynomial $P_n(x)$ of degree n and leading coefficient p is

$$\text{Disc}_x(P; q) = p^{2n-2} q^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (q^{-1/2} x_i - q^{1/2} x_j)(q^{1/2} x_i - q^{-1/2} x_j) \quad (1)$$

where x_i , $1 \leq i \leq n$, are the roots of $P_n(x)$. This q -discriminant is 0 if and only if a quotient of roots x_i/x_j equals q . As $q \rightarrow 1$, this q -discriminant becomes the ordinary discriminant which is denoted by $\text{Disc}_x P(x)$. For the study of resultants and ordinary discriminants and their various formulas, see [1], [2], [9], and [11].

We will see that the concept of q -discriminant is useful in proving connections between the root distribution of a sequence of polynomials $H_m(z)$ and the discriminant of the denominator of its generating function $\text{Disc}_t D(t, z)$. We will show in the three cases mentioned above that the roots of $H_m(z)$ lie on a portion of a real algebraic curve (see Theorem 1, Theorem 3, and Theorem 5). For the study of sequences of polynomials whose roots approach fixed curves, see [5, 6, 7]. Other studies of the limits of zeros of polynomials satisfying a linear homogeneous recursion whose coefficients are polynomials in z are given in [3, 4]. The q -discriminant will appear as the quotient q of roots in t of $D(t, z)$. One advantage of looking at the quotients of roots is that, at least in the three cases above, although the roots of $H_m(z)$ lie on a curve depending on $A(z)$ and $B(z)$, the quotients of roots $t = t(z)$ of $D(t, z)$ lie on a fixed curve independent of these two polynomials. We will show that this independent curve is the unit circle in the quadratic case and two peculiar curves (see Figures 1 and 2 in Sections 3 and 4) in the cubic and quartic cases. From computer experiments, this curve looks more complicated in the quintic case $D(z, t) = 1 + B(z)t + A(z)t^5$ (see Figure 3 in Section 4).

As an application of these theorems, we will consider an example where $D(t, z) = 1 + (z^2 - 2z + a)t + z^2 t^2$ and $a \in \mathbb{R}$. We will see that the roots of $H_m(z)$ lie either on portions of the circle of radius \sqrt{a} or real intervals depending on the value a compared to the critical values 0 and 4. Also, the endpoints of the curves where the roots of $H_m(z)$ lie are roots of $\text{Disc}_t D(t, z)$. Interestingly, the critical values 0 and 4 are roots of the double discriminant $\text{Disc}_z \text{Disc}_t D(t, z) = 4096a^3(a - 4)$.

2 The quadratic denominator

In this section, we will consider the root distribution of $H_m(z)$ when the denominator of the generating function is $D(t, z) = 1 + B(z)t + A(z)t^2$.

Theorem 1 *Let $H_m(z)$ be a sequence of polynomials whose generating function is*

$$\sum H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^2}$$

where $A(z)$ and $B(z)$ are polynomials in z with complex coefficients. The roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on the curve \mathcal{C}_2 defined by

$$\Im \frac{B^2(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq \Re \frac{B^2(z)}{A(z)} \leq 4,$$

and are dense there as $m \rightarrow \infty$.

Proof Suppose z_0 is a root of $H_m(z)$ which satisfies $A(z_0) \neq 0$. Let $t_1 = t_1(z_0)$ and $t_2 = t_2(z_0)$ be the roots of $D(t, z_0)$. If $t_1 = t_2$ then $\text{Disc}_t D(t, z_0) = B^2(z_0) - 4A(z_0) = 0$. In this case z_0 belongs to \mathcal{C}_2 , and we only need to consider the case $t_1 \neq t_2$. By partial fractions, we have

$$\begin{aligned} \frac{1}{D(t, z_0)} &= \frac{1}{A(z_0)(t - t_1)(t - t_2)} \\ &= \frac{1}{A(z_0)(t_1 - t_2)} \left(\frac{1}{t - t_1} - \frac{1}{t - t_2} \right) \\ &= \frac{1}{A(z_0)} \sum_{m=0}^{\infty} \frac{t_1^{m+1} - t_2^{m+1}}{(t_1 - t_2)t_1^{m+1}t_2^{m+1}} t^n. \end{aligned} \quad (2)$$

Thus if we let $t_1 = qt_2$ then q is an $(m+1)$ -st root of unity and $q \neq 1$. By the definition of q -discriminant in (1), q is a root of $\text{Disc}_t(D(t, z_0); q)$ which equals

$$q(B^2(z_0) - (q + q^{-1} + 2)A(z_0)).$$

This implies that

$$\frac{B^2(z_0)}{A(z_0)} = q + q^{-1} + 2.$$

Thus $z_0 \in \mathcal{C}_2$ since q is an $(m+1)$ -th root of unity.

The map $B^2(z)/A(z)$ maps an open neighborhood U of a point on \mathcal{C}_2 onto an open set which contains a point $2\Re q + 2$, where q is an $(m+1)$ -th root of unity, when m is large. From (2), there is a solution of $H_m(z)$ in U . The density of the roots of $H_m(z)$ follows.

Example We consider an example in which the generating function of $H_m(z)$ is given by

$$\frac{1}{z^2 t^2 + (z^2 - 2z + a)t + 1} = \sum_{m=0}^{\infty} H_m(z) t^m$$

where $a \in \mathbb{R}$. Let $z = x + iy$. We exhibit the three possible cases for the root distribution of $H_m(z)$ depending on a :

1. If $a \leq 0$, the roots of $H_m(z)$ lie on the two real intervals defined by

$$(x^2 + a)(x^2 - 4x + a) \leq 0.$$

2. If $0 < a \leq 4$, the roots of $H_m(z)$ can lie either on the half circle $x^2 + y^2 = a$, $x \geq 0$, or on the real interval defined by $x^2 - 4x + a \leq 0$.

3. If $a > 4$, the roots of $H_m(z)$ lie on two parts of the circle $x^2 + y^2 = a$ restricted by $0 \leq x \leq 2$.

Indeed, by complex expansion, we have

$$\Im \frac{B^2(z)}{A(z)} = \frac{2y(x^2 + y^2 - a)P}{(x^2 + y^2)^2} \quad \text{and} \quad \Re \frac{B^2(z)}{A(z)} = \frac{P^2 - Q^2}{(x^2 + y^2)^2},$$

where

$$P = ax - 2x^2 + x^3 - 2y^2 + xy^2 \quad \text{and} \quad Q = y(x^2 + y^2 - a).$$

Theorem 1 yields three cases: $y = 0$, $x^2 + y^2 - a = 0$ or $P = 0$. Since $\Re(B^2(z)/A(z)) \geq 0$, all these cases give $Q = 0$. We note that if $x^2 + y^2 - a = 0$ then the condition $\Re(B^2(z)/A(z)) \leq 4$ reduces to

$$x(a + x^2 + y^2)(ax - 4x^2 + x^3 - 4y^2 + xy^2) = 4a^2x(x - 2) \leq 0. \quad (3)$$

Suppose $a \leq 0$. Then the condition $Q = 0$ implies that the roots of $H_m(z)$ are real. The condition $\Re(B^2(z)/A(z)) \leq 4$ becomes

$$(x^3 - 2x^2 + ax)^2 - 4x^4 = x^2(x^2 + a)(x^2 - 4x + a) \leq 0. \quad (4)$$

Suppose $0 < a \leq 4$. The roots of $H_m(z)$ lie either on the half circle $x^2 + y^2 - a = 0$, $x \geq 0$ (from the inequality (3)), or on the real interval given by $x^2 - 4x + a \leq 0$ (from the inequality (4)). If $a > 4$ then the roots of $H_m(z)$ lie on the two parts of the circle $x^2 + y^2 - a = 0$ restricted by $0 \leq x \leq 2$ (from the inequality (3)).

We notice that in this example, the inequality $\Re(B^2(z)/A(z)) \leq 4$ gives the endpoints of the curves where the roots of $H_m(z)$ lie. Thus, these endpoints are roots of $\text{Disc}_t(1 + B(z)t + A(z)t^2) = B^2(z) - 4A(z)$. Moreover the critical values of a , which are 0 and 4, are roots of the double discriminant of the denominator

$$\text{Disc}_z \text{Disc}_t(1 + (z^2 - 2z + a)t + z^2t^2) = 4096a^3(a - 4).$$

This comes from the fact that the endpoints of the fixed curves containing the roots of $H_m(z)$ are the roots of $\text{Disc}_t(1 + (z^2 - 2z + a)t + z^2t^2)$. When this discriminant has a double root as a polynomial in z , some two endpoints of the fixed curves coincide. That explains the change in the shape of the root distribution.

3 The cubic denominator

In this section we show that in the cubic case $D(t, z) = 1 + B(z)t + A(z)t^3$, the roots of $H_m(z)$ lie on a portion of a real algebraic curve. As we see in the proof of Theorem 1, we can first consider the distribution of the quotients of roots $q = t_i/t_j$ of $D(t, z)$, and then we can relate to the root distribution of $H_m(z)$ using the q -discriminant. While in the previous section this quotient lies on the unit circle, in this section we show that this quotient lie on the curve in Figure 1.

Lemma 2 Suppose $\zeta_1, \zeta_2 \neq 0$ are complex numbers such that $1/\zeta_1 + 1/\zeta_2 + 1 = 0$ and

$$\frac{\zeta_1^{m+1} - 1}{\zeta_1 - 1} = \frac{\zeta_2^{m+1} - 1}{\zeta_2 - 1}. \quad (5)$$

Then ζ_1 and ζ_2 lie on the union $C_1 \cup C_2 \cup C_3$ where the Cartesian equations of C_1 , C_2 and C_3 are given by

$$\begin{aligned} C_1 & : (x+1)^2 + y^2 = 1, x \leq -\frac{1}{2}, \\ C_2 & : x = -\frac{1}{2}, -\frac{\sqrt{3}}{2} \leq y \leq \frac{\sqrt{3}}{2}, \\ C_3 & : x^2 + y^2 = 1, x \geq -\frac{1}{2}, \end{aligned}$$

and are dense there as $m \rightarrow \infty$.

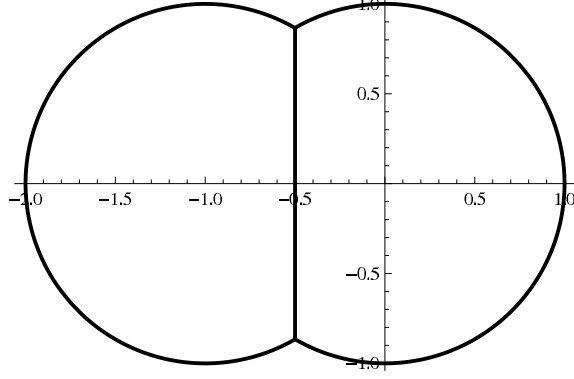


Figure 1: Distribution of the quotients of the roots of the cubic denominator

Proof We can rewrite (5) as

$$\sum_{k=0}^m \zeta_1^k = \sum_{k=0}^m \zeta_2^k$$

where we can replace ζ_2 by $-\zeta_1/(\zeta_1 + 1)$. By multiplying both sides by $(\zeta_1 + 1)^m$, we note that there are at most $2m - 2$ solutions $\zeta = \zeta_1 \neq 0, -2$ counting multiplicity. Let $m = 3n + k$ where $k = 1, 2, 3$. From implicit differentiation, we can check that the equation (5) has roots at $e^{2\pi i/3}, e^{4\pi i/3}$ with multiplicity $k - 1$. After subtracting this number of roots from $2m - 2$, we conclude that there are at most $6n$ roots $\zeta \neq 0, -2, e^{2\pi i/3}, e^{4\pi i/3}$. We first show that if $\zeta \neq -2$ is a root, then so is $-\zeta - 1$. From the two equations in the hypothesis, we note that $\zeta \neq 0, -1$ and

$$\sum_{k=0}^m \zeta^k = \sum_{k=0}^m \left(-\frac{\zeta}{\zeta + 1} \right)^k.$$

Subtracting 1, then dividing by ζ and multiplying both sides by $(\zeta + 1)^m$, we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{m-1} \zeta^k (\zeta + 1)^m + \sum_{k=0}^{m-1} (\zeta + 1)^{m-k-1} (-\zeta)^k \\ &= \sum_{k=0}^{m-1} \zeta^k (\zeta + 1)^{m-k-1} ((\zeta + 1)^{k+1} - (-1)^{k+1}) \\ &= (\zeta + 2) \sum_{k=0}^{m-1} \zeta^k (\zeta + 1)^{m-k-1} \sum_{i=0}^k (\zeta + 1)^{k-i} (-1)^i \\ &= (\zeta + 2) \sum_{k=0}^{m-1} \sum_{i=0}^k \zeta^k (-\zeta - 1)^{m-1-i}. \end{aligned}$$

By interchanging the summation and reversing the index of summation we obtain

$$\begin{aligned} \sum_{k=0}^{m-1} \sum_{i=0}^k \zeta^k (-\zeta - 1)^{m-1-i} &= \sum_{i=0}^{m-1} \sum_{k=i}^{m-1} \zeta^k (-\zeta - 1)^{m-1-i} \\ &= \sum_{i=0}^{m-1} \sum_{k=0}^i \zeta^{m-1-k} (-\zeta - 1)^i. \end{aligned}$$

Hence we have symmetry between ζ and $-1 - \zeta$ in the two double summations.

Our goal is to show that the number of roots $\zeta \neq 0, -2, e^{2\pi i/3}, e^{4\pi i/3}$ on $C_1 \cup C_2 \cup C_3$ is at least $6n$, counting multiplicities. Then all roots will lie on $C_1 \cup C_2 \cup C_3$ since we have at most $6n$ roots $\zeta \neq 0, -2, e^{2\pi i/3}, e^{4\pi i/3}$. By the symmetry of roots mentioned above, if $\zeta \neq -2$ is a solution in C_1 then $(-1 - 1/\zeta, -\zeta - 1)$ is a solution in $C_2 \times C_3$. Hence there is a bijection between roots in $C_1 \setminus \{-2\}$, C_2 and C_3 . Thus if $C_1 \setminus \{e^{2\pi i/3}, e^{4\pi i/3}\}$ contains at least $2n + 1$ roots then all of the roots lie on $C_1 \cup C_2 \cup C_3$. Let $\zeta = \zeta_1$ be a root on $C_1 \setminus \{e^{2\pi i/3}, e^{4\pi i/3}\}$. Then the equation $1/\zeta_1 + 1/\zeta_2 + 1 = 0$ gives $\zeta_2 = \bar{\zeta}$. Thus (5) gives

$$\Im \frac{\zeta^{m+1} - 1}{\zeta - 1} = 0.$$

Write $\zeta = re^{i\theta}$ where $r = -2 \cos \theta$, $\cos \theta \leq -1/2$. Then complex expansion yields

$$r^{m+2} \sin m\theta - r^{m+1} \sin(m+1)\theta + r \sin \theta = 0.$$

Divide r , replace r by $-2 \cos \theta$ and combine the first two terms to obtain

$$\begin{aligned} 0 &= (-1)^{m+1} 2^m \cos^m \theta (2 \sin m\theta \cos \theta + \sin(m+1)\theta) + \sin \theta \\ &= (-1)^{m+1} 2^m \cos^m \theta (2 \sin(m+1)\theta - 2 \cos m\theta \sin \theta + \sin(m+1)\theta) + \sin \theta \\ &= (-1)^{m+1} 2^m \cos^m \theta (2 \sin(m+1)\theta + \sin m\theta \cos \theta - \cos m\theta \sin \theta) + \sin \theta \\ &= (-1)^{m+1} 2^m \cos^m \theta (2 \sin(m+1)\theta + \sin(m-1)\theta) + \sin \theta. \end{aligned}$$

We note that the right side has different signs if $\sin(m+1)\theta = 1$ and $\sin(m+1)\theta = -1$. Thus we can apply the Intermediate Value Theorem on several intervals whose boundaries are the solutions of $\sin(m+1)\theta = \pm 1$. The equations $\sin(m+1)\theta = \pm 1$ give

$$(m+1)\theta = \pm \frac{\pi}{2} + 2j\pi.$$

The condition $2\pi/3 < \theta < 4\pi/3$ and the fact that $m = 3n + k$, $k = 1, 2, 3$, yield

$$n + \frac{k+1}{3} \pm \frac{1}{4} < j < 2n + \frac{2(k+1)}{3} \pm \frac{1}{4}.$$

If $k = 1$, we have at least $2n + 1$ roots coming from $2n + 1$ intervals formed by the $2n + 2$ points

$$\frac{2j\pi \pm \pi/2}{m+1},$$

where $n < j \leq 2n + 1$. If $k = 2$, we have at least $2n + 1$ roots coming from $2n + 1$ intervals formed by the $2n + 2$ points

$$\left\{ \frac{2j - \pi/2}{m+1} : n+1 \leq j < 2n+2 \right\} \cup \left\{ \frac{2j + \pi/2}{m+1} : n+1 < j \leq 2n+2 \right\}.$$

If $k = 3$, we have at least $2n + 1$ roots coming from $2n + 1$ intervals formed by the $2n + 2$ points

$$\frac{2j\pi \pm \pi/2}{m+1},$$

where $n + 1 < j < 2n + 2$. The density follows from the distribution of $2n + 1$ roots mentioned above. The lemma follows.

Theorem 3 *Let $H_m(z)$ be a sequence of polynomials whose generating function is*

$$\sum H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^3}$$

where $A(z)$ and $B(z)$ are polynomials in z with complex coefficients. The roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on the curve \mathcal{C}_3 defined by

$$\Im \frac{B^3(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq -\Re \frac{B^3(z)}{A(z)} \leq \frac{3^3}{2^2},$$

and are dense there as $m \rightarrow \infty$.

Proof For a little simplification, we consider the roots of $H_{m-1}(z)$. Let z_0 be a root of $H_{m-1}(z)$ which satisfies $A(z_0) \neq 0$. Let $t_1 = t_1(z_0)$, $t_2 = t_2(z_0)$ and $t_3 = t_3(z_0)$ be the roots of $D(t, z_0) = 1 + B(z_0)t + A(z_0)t^3$. It suffices to consider $\text{Disc}_t(D(t, z_0)) = -4A(z_0)B^3(z_0) - 27A^2(z_0) \neq 0$. By partial fractions, the function $1/D(t, z_0)$ is

$$\frac{1}{A(z_0)(t_1 - t_2)(t_1 - t_3)(t - t_1)} + \frac{1}{A(z_0)(t_2 - t_1)(t_2 - t_3)(t - t_2)} + \frac{1}{A(z_0)(t_3 - t_1)(t_3 - t_2)(t - t_3)}.$$

We expand $1/(t - t_i)$ using geometric series and write the expression above as

$$\sum_{m=1}^{\infty} \frac{t_1^{m+1}t_2^m - t_1^m t_2^{m+1} - t_1^{m+1}t_3^m + t_2^{m+1}t_3^m + t_1^m t_3^{m+1} - t_2^m t_3^{m+1}}{A(z_0)t_1^m t_2^m t_3^m (t_1 - t_2)(t_1 - t_3)(t_2 - t_3)} t^{m-1}.$$

Since z_0 is a root of $H_{m-1}(z)$, we have

$$t_1^{m+1}t_2^m - t_1^m t_2^{m+1} - t_1^{m+1}t_3^m + t_2^{m+1}t_3^m + t_1^m t_3^{m+1} - t_2^m t_3^{m+1} = 0.$$

We divide this equation by t_3^{2m+1} and let $q = q_1 = t_1/t_3$, $q_2 = t_2/t_3$ to obtain

$$q_1^{m+1}q_2^m - q_1^m q_2^{m+1} - q_1^{m+1} + q_2^{m+1} + q_1^m - q_2^m = 0$$

where $q_1 + q_2 + 1 = 0$ since $t_1 + t_2 + t_3 = 0$. The equation can be written as

$$q_1^m q_2^m (q_1 - q_2) - q_1^m (q_1 - 1) + q_2^m (q_2 - 1) = 0.$$

Since $q_1^m q_2^m (q_1 - q_2) = q_1^m q_2^m (q_1 - 1) - q_1^m q_2^m (q_2 - 1)$ and $q_1, q_2 \neq 0, 1$, this equation becomes

$$\frac{q_1^m - 1}{q_1^m (q_1 - 1)} = \frac{q_2^m - 1}{q_2^m (q_2 - 1)}.$$

Let $\zeta_1 = 1/q_1$ and $\zeta_2 = 1/q_2$ and add 1 to both sides. Then

$$\frac{\zeta_1^{m+1} - 1}{\zeta_1 - 1} = \frac{\zeta_2^{m+1} - 1}{\zeta_2 - 1}.$$

Thus ζ_1 and ζ_2 (and also q_1 and q_2) lie on the curve given in Lemma 2. Since q_1 and q_2 are given by quotients of two roots, they are roots of the q -discriminant given by

$$\text{Disc}_t(D(t, z_0); q) = -B^3(z_0)A(z_0)q^2(1+q)^2 - A^2(z_0)(1+q+q^2)^3.$$

This gives

$$\frac{B^3(z_0)}{A(z_0)} = -\frac{(1+q+q^2)^3}{q^2(1+q)^2}.$$

It remains to show that the map

$$f(q) = -\frac{(1+q+q^2)^3}{q^2(1+q)^2}$$

maps the curve in Figure 1 to the real interval $[-27/4, 0]$. Let q be a point on the this curve. We note that

$$f(q) = f(-1-q) = -\frac{(q^{-1}+1+q)^3}{q^{-1}+2+q}.$$

Since q lies on the curve in Figure 1, we have the three possible cases $\bar{q} = -1-q$, $|q| = 1$ or $|-1-q| = 1$. In the first case, $\Im f(q) = 0$ since $f(q) = \overline{f(q)}$. In the second and third cases, $\Im f(q) = 0$ since $q + q^{-1} \in \mathbb{R}$ and $f(q) = f(-1-q)$. Furthermore, $f(q)$ attains its minimum and maximum when $q = 1$ and $q = e^{2\pi i/3}$ respectively. The density of the roots of $H_m(z)$ follows from similar arguments as in the proof of Theorem 1.

4 The quartic denominator

In this section, we will show that in the case $D(t, z) = 1 + B(z) + A(z)t^4$ the roots of $H_m(z)$ lie on a portion of a real algebraic curve. Similar to the approach in the previous sections, we first consider the distribution of the quotients of roots of $D(t, z)$. Before looking at these quotients, let us recall that the Chebyshev polynomial of the second kind $U_m(z)$ is

$$U_m(z) = \frac{\sin(m+1)\theta}{\sin \theta}$$

where

$$z = \cos \theta.$$

Suppose $z_1, z_2 \in \mathbb{C}$ such that $|z_1| = |z_2|$. Let $e^{2i\theta} = z_1/z_2$ and $z = \cos \theta$. If k is a positive integer then

$$\begin{aligned} \frac{z_1^k - z_2^k}{z_1 - z_2} &= (z_1 z_2)^{(k-1)/2} \frac{(z_1/z_2)^{m/2} - (z_2/z_1)^{m/2}}{(z_1/z_2)^{1/2} - (z_2/z_1)^{1/2}} \\ &= (z_1 z_2)^{(k-1)/2} U_m(z). \end{aligned} \tag{6}$$

By analytic continuation, we can extend this identity to any pair of complex numbers z_1 and z_2 with

$$2z = \left(\frac{z_1}{z_2}\right)^{1/2} + \left(\frac{z_2}{z_1}\right)^{1/2}.$$

Lemma 4 Suppose z_0 is a root of $H_m(z)$ and $q = q(z_0)$ is a quotient of two roots in t of $1 + B(z_0)t + A(z_0)t^4$. Then the set of all such quotients belongs to the curve depicted in Figure 2, where the Cartesian equation of the quartic curve on the left is

$$1 + 2x + 2x^2 + 2x^3 + x^4 - 2y^2 + 2xy^2 + 2x^2y^2 + y^4 = 0,$$

and the curve on the right is the unit circle with real part at least $-1/3$. All such quotients are dense on this curve as $m \rightarrow \infty$.

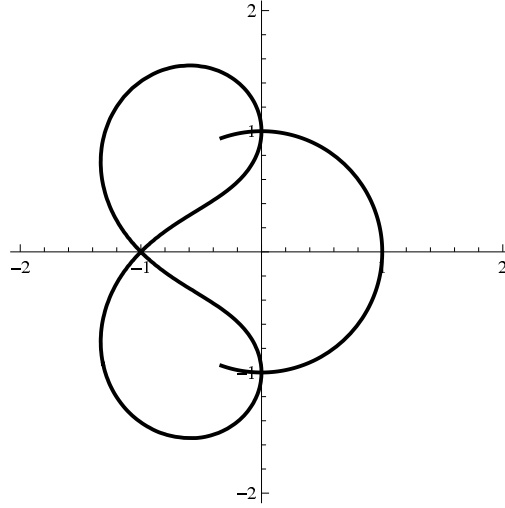


Figure 2: Distribution of the quotients of the roots of the quartic denominator

Proof For each $z_0 \in \mathbb{C}$, let $t_1 = t_1(z_0)$, $t_2 = t_2(z_0)$, $t_3 = t_3(z_0)$, and $t_4 = t_4(z_0)$ be the roots of the denominator $1 + B(z_0)t + A(z_0)t^4$. By partial fractions, we have

$$\begin{aligned} \frac{1}{1 + B(z_0)t + A(z_0)t^4} &= \frac{1}{A(z_0)(t - t_1)(t - t_2)(t - t_3)(t - t_4)} \\ &= \sum_{m=0}^{\infty} H_m(z_0)t^m, \end{aligned}$$

where

$$\begin{aligned} A(z_0)H_m(z_0) &= \frac{1}{t_1^{m+1}(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)} + \frac{1}{t_2^{m+1}(t_2 - t_1)(t_2 - t_3)(t_2 - t_4)} \\ &\quad + \frac{1}{t_3^{m+1}(t_3 - t_1)(t_3 - t_2)(t_3 - t_4)} + \frac{1}{t_4^{m+1}(t_4 - t_1)(t_4 - t_2)(t_4 - t_3)}. \end{aligned}$$

Let $q_1 = t_1/t_4$, $q_2 = t_2/t_4$, $q_3 = t_3/t_4$. For a little reduction in the powers of q_i , $1 \leq i \leq 3$, we will consider the roots of the polynomial $H_{m-2}(z)$. We put all terms of $A(z_0)H_{m-2}(z_0)$ over a common denominator and then divide the numerator by t_4^{3m} . The condition $H_{m-2}(z_0) = 0$ implies

$$\begin{aligned} 0 = & q_1^{m+1}(-q_2^{m-1}q_3^{m-1}(q_2 - q_3) + q_2^m - q_3^m - q_2^{m-1} + q_3^{m-1}) \\ & + q_1^m(q_2^{m-1}q_3^{m-1}(q_2^2 - q_3^2) - q_2^{m+1} + q_3^{m+1} + q_2^{m-1} - q_3^{m-1}) \\ & + q_1^{m-1}(-q_2^mq_3^m(q_2 - q_3) + q_2^{m+1} - q_3^{m+1} - q_2^m + q_3^m) \\ & + q_2^{m-1}q_3^{m-1}(q_2 - q_3) - q_2^{m-1}q_3^{m-1}(q_2^2 - q_3^2) + q_2^mq_3^m(q_2 - q_3). \end{aligned} \quad (7)$$

The fact

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= 0 \\ t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 &= 0 \end{aligned}$$

gives

$$q_2 + q_3 = -1 - q_1 \quad \text{and} \quad q_2q_3 = q_1^2 + q_1 + 1.$$

From the symmetric reductions, the right side of (7), after being divided by $q_2 - q_3$, is a polynomial in q_1 of degree $3m - 1$. We used a computer algebra system to check for the root distribution of this polynomial in the case $m \leq 5$. We now assume that $m \geq 6$. We will show that the number of roots q_1 lying on the two curves in Figure 2 is at least $3m - 1$. The first step is to show that if the set of q_1 belongs to the unit circle with $\Re q_1 \geq -1/3$ and is dense there as $m \rightarrow \infty$ then the set of q_2 and q_3 belongs to the quartic curve given in the lemma and is dense on this quartic curve as $m \rightarrow \infty$. Then we will find the number of roots q_1 on the unit circle with $\Re q_1 \geq -1/3$.

Suppose $q_1 = e^{i\pi\theta}$ lies on the unit circle and $1 \geq \cos \theta \geq -1/3$. We note that q_2 and q_3 are the two roots of the equation

$$f(q) := q^2 + (1 + q_1)q + q_1^2 + q_1 + 1 = 0.$$

Thus the quadratic formula gives

$$q = \frac{-1 - e^{i\theta} \pm ie^{i\theta/2}\sqrt{6\cos\theta + 2}}{2}.$$

Splitting the real and imaginary parts of the function on the left side, we leave it to the reader to check that this function maps the interval $1 \geq \cos \theta \geq -1/3$ to the quartic curve

$$1 + 2x + 2x^2 + 2x^3 + x^4 - 2y^2 + 2xy^2 + 2x^2y^2 + y^4 = 0.$$

We now compute the number of roots $q_1 = e^{i\pi\theta}$ with $\cos \theta \geq -1/3$. We first consider $q_1 \neq \pm i, 1$. Let

$$2\zeta = \left(\frac{q_2}{q_3}\right)^{1/2} + \left(\frac{q_3}{q_2}\right)^{1/2}.$$

Equation (6) gives

$$\frac{q_2^m - q_3^m}{q_2 - q_3} = (q_2q_3)^{(m-1)/2}U_{m-1}(\zeta)$$

where

$$\zeta^2 = \frac{1}{4} \frac{(q_2 + q_3)^2}{q_2 q_3} = \frac{(q_1 + 1)^2}{4(q_1^2 + q_1 + 1)} = \frac{1}{4(2 \cos \theta + 1)} + \frac{1}{4} \in \mathbb{R}. \quad (8)$$

We divide (7) by $q_2 - q_3$ and rewrite it in terms of Chebyshev polynomials:

$$\begin{aligned} 0 &= U_m(\zeta)(-q_1^m + q_1^{m-1})(q_1^2 + q_1 + 1)^{m/2} \\ &\quad + U_{m-1}(\zeta)(q_1^{m+1} - q_1^{m-1})(q_1^2 + q_1 + 1)^{(m-1)/2} \\ &\quad + U_{m-2}(\zeta)(-q_1^{m+1} + q_1^m)(q_1^2 + q_1 + 1)^{(m-2)/2} \\ &\quad + (q_1^2 + q_1 + 1)^{m-1}(-3q_1^{m+1} - 2q_1^m - q_1^{m-1} + q_1^2 + 2q_1 + 3). \end{aligned}$$

We divide this equation by $q_1^{(3m-2)/2}(1 - q_1)(q_1 + q_1^{-1} + 1)^{m/2}$ and write $(q_1^m - 1)/(q_1 - 1)$ in terms of Chebyshev polynomials. We obtain

$$\begin{aligned} 0 &= U_m(\zeta) + 2\zeta U_{m-1}(\zeta) + U_{m-2}(\zeta)/(q_1 + q_1^{-1} + 1) \\ &\quad + (q_1 + q_1^{-1} + 1)^{m/2-1} (3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)), \end{aligned}$$

where

$$\xi^2 = \frac{(q_1 + 1)^2}{4q_1} = \frac{2 \cos \theta + 2}{4}. \quad (9)$$

Finally, from (8) we can replace $1/(q_1 + q_1^{-1} + 1)$ by $(4\zeta^2 - 1)$ and use the recurrence definition of the Chebyshev polynomials to rewrite this equation in the symmetric form below:

$$\begin{aligned} 0 &= (4\zeta^2 - 1)^{(m-2)/4} (3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta)) \\ &\quad + (4\xi^2 - 1)^{(m-2)/4} (3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)). \end{aligned} \quad (10)$$

From this symmetric form, the right expression remains the same if we interchange ζ and ξ or if we interchange $\cos \theta$ and $-\cos \theta/(2 \cos \theta + 1)$ (from (8) and (9)). Thus the numbers of roots q_1 are the same in the two cases $0 < \cos \theta < 1$ and $-1/3 < \cos \theta < 0$. It is sufficient to count the number of roots $0 < \cos \theta < 1$ or $1/2 < \xi^2 < 1$. Let $\cos \alpha = \xi$ and $U_m(\xi) = \sin(m+1)\alpha/\sin \alpha$ where $-\pi/4 < \alpha < \pi/4$, $\alpha \neq 0$. The idea is to show that in this case the summand

$$(4\xi^2 - 1)^{(m-2)/4} (3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)) \quad (11)$$

dominates the right expression of (10). Since ζ^2 and ξ^2 in (10) are real numbers and the Chebyshev polynomials in this equation are either even or odd, we can apply the Intermediate Value Theorem. We note that (11) has different signs when $\sin(m+1)\alpha = 1$ and when $\sin(m+1)\alpha = -1$. Suppose $\sin(m+1)\alpha = \pm 1$ and $-\pi/4 < \alpha < \pi/4$. Since

$$4\zeta^2 - 1 = \frac{1}{1 + 2 \cos \theta} < 1,$$

it suffices to show

$$\left| (4\xi^2 - 1)^{(m-2)/2} (3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)) \right| \geq |3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta)|. \quad (12)$$

Let $\zeta = \cos \beta$. Using the fact that $1/3 < \zeta^2 < 1/2$ and $U_m(\zeta) = \sin(m+1)\beta / \sin \beta$, we obtain the following upper bound for the right hand side of (12):

$$|3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta)| \leq 6\sqrt{2}.$$

Since

$$\alpha = \frac{\pi}{4} \frac{(4k \pm 2)}{m+1} \quad (13)$$

where $k \in \mathbb{Z}$ and $-\pi/4 < \alpha < \pi/4$, we have

$$|\alpha| \leq \frac{\pi}{4} \left(1 - \frac{1}{m+1}\right).$$

Thus

$$\cos \alpha \geq \frac{\sqrt{2}}{2} \left(\cos \frac{\pi}{4(m+1)} + \sin \frac{\pi}{4(m+1)} \right).$$

This inequality and (9) give

$$2 \cos \theta = 4 \cos^2 \alpha - 2 \geq 4 \sin \frac{\pi}{2(m+1)}. \quad (14)$$

From the definition of the Chebyshev polynomial, we have

$$\begin{aligned} U_{m-2}(\xi) &= \frac{\sin(m-1)\alpha}{\sin \alpha} \\ &= \frac{\sin(m+1)\alpha \cos 2\alpha - \cos(m+1)\alpha \sin 2\alpha}{\sin \alpha} \\ &= \frac{\sin(m+1)\alpha \cos 2\alpha}{\sin \alpha}. \end{aligned}$$

With similar computations for $U_{m-4}(\xi)$, we obtain

$$|3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)| = \frac{|\sin(m+1)\alpha| |3 + 2 \cos 2\alpha + \cos 4\alpha|}{|\sin \alpha|}.$$

Since $\sin(m+1)\alpha = \pm 1$ and $\cos 2\alpha \geq 0$, the right side is at least $2\sqrt{2}$. We combine this with (14) to have

$$\begin{aligned} \left| (2 \cos \theta + 1)^{(m-2)/2} (3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)) \right| &\geq \left(1 + 4 \sin \frac{\pi}{2(m+1)} \right)^{(m-2)/2} 2\sqrt{2} \\ &\geq 6\sqrt{2} \end{aligned}$$

when $m \geq 6$. The inequality (12) follows. By the Intermediate Value Theorem, we have at least one root when $\sin(m+1)\alpha$ changes between -1 and 1 with $-\pi/4 < \alpha < \pi/4$. From the formula (13), the number of roots q_1 when $0 < \cos \theta < 1$ is at least $2(\lfloor (m-2)/4 \rfloor)$. By symmetry, the number of roots $q_1 \neq \pm i, 1$ with $\Re q_1 > -1/3$ on the unit circle is at least $4(\lfloor (m-2)/4 \rfloor)$. Note that each of these roots gives two more roots q_2 and q_3 on the quartic curve.

It remains to check the multiplicities of $q_1 = \pm i, 1$ in the equation (7). We note that this equation has a root $q_1 = 1$ with multiplicity at least 1. In the case $q_1 = 1$ we obtain four more roots q_2, q_2^{-1}, q_3 , and q_3^{-1} . We now consider the case $q_1 = \pm i$. The equation $q^2 + (1 + q_1)q + q_1^2 + q_1 + 1 = 0$ where $q = q_2, q_3$ gives $(q_2, q_3) = (-1, i)$ or $(q_2, q_3) = (i, -1)$ when $q_1 = i$ and $(q_2, q_3) = (-1, -i)$ or $(q_2, q_3) = (-i, -1)$ when $q_1 = -i$. Hence each of the roots $q_1 = \pm i$ gives us another root at -1 with the same multiplicity. To check the multiplicities at $q_1 = \pm i$, we need to differentiate the equation (7) with respect to q_1 . We obtain its derivatives by applying implicit differentiation to the equation $q^2 + (1 + q_1)q + q_1^2 + q_1 + 1 = 0$. After substituting $q_1 = \pm i$ in (7) and its derivatives, we see that the multiplicity of $\pm i$ is

$$\begin{cases} 2 & \text{if } m = 4k \\ 3 & \text{if } m = 4k + 1 \\ 0 & \text{if } m = 4k + 2 \\ 1 & \text{if } m = 4k + 3 \end{cases}.$$

The table below tabulates the $3m - 1$ roots of (7).

	$m = 4k$	$m = 4k + 1$	$m = 4k + 2$	$m = 4k + 3$
$q_1 = e^{i\theta}, \Re q_1 > -1/3, q_1 \neq \pm i, 1$	$3(4k - 4)$	$3(4k - 4)$	$12k$	$12k$
$q_1 = 1$	5	5	5	5
$q_1 = \pm i$	6	9	0	3
Total	$12k - 1$	$12k + 2$	$12k + 5$	$12k + 8$

All the roots counted on the table lie on the curves given in the lemma. The number of roots counted equals the number of possible roots which is $3m - 1$. Also, as a consequence of the Intermediate Value Theorem applied to the intervals formed by $\sin(m + 1)\alpha = \pm 1$, the roots q_1 are dense on the portion of the unit circle with real part at least $-1/3$. The lemma follows.

Theorem 5 *Let $H_m(z)$ be a sequence of polynomials whose generating function is*

$$\sum H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^4}$$

where $A(z)$ and $B(z)$ are polynomials in z with complex coefficients. The roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on the curve \mathcal{C}_4 defined by

$$\Im \frac{B^4(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq \Re \frac{B^4(z)}{A(z)} \leq \frac{4^4}{3^3},$$

and are dense there as $m \rightarrow \infty$.

Proof From the definition of q -discriminant in (1), we have

$$\text{Disc}_t(1 + B(z)t + A(z)t^4; q) = -A^2(z)B^4(z)q^3(1 + q + q^2)^3 + A^3(z)(1 + q + q^2 + q^3)^4.$$

If q is a quotient of two roots of $1 + B(z)t + A(z)t^4$, then

$$\frac{B^4(z)}{A(z)} = \frac{(1 + q + q^2 + q^3)^4}{q^3(1 + q + q^2)^3}.$$

Let $f(q)$ be the function on the right side. We note that $f(q)$ maps $q_1 = e^{i\theta}$ with $\Re q_1 \geq -1/3$ to the real interval $[0, 4^4/3^3]$ since

$$f(q_1) = \frac{(q_1^{3/2} + q_1^{-3/2} + q_1^{1/2} + q_1^{-1/2})^4}{(q_1 + q_1^{-1} + 1)^3}.$$

If q is a point on the quartic curve in Lemma 4 then q and q_1 are related by

$$q_1^2 + q^2 + q_1q + q_1 + q + 1 = 0.$$

Multiplying this equation by $q_1 - q$, we obtain

$$q_1^3 + q_1^2 + q_1 = q^3 + q^2 + q.$$

Thus by the definition of $f(q)$, we have $f(q) = f(q_1)$. Since

$$\text{Disc}_t(1 + B(z)t + A(z)t^4) = -3^3 A^2(z)B^4(z) + 4^4 A^3(z),$$

the roots of $H_m(z)$ lie on the curve \mathcal{C}_4 . The density of these roots follows from arguments similar to those in the proof of Theorem 1.

Remark: One may try to find the root distribution of $H_m(z)$ in the case $D(t, z) = 1 + B(z)t + A(z)t^5$. From computer experiments, the distribution of the quotients of roots of $D(t, z)$ in the case $m = 50$ is given in the figure below.

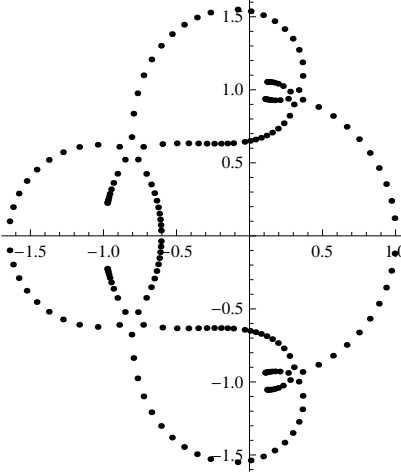


Figure 3: Distribution of the quotients of roots of the quintic denominator

We end this paper with the following conjecture.

Conjecture 6 *Let $H_m(z)$ be a sequence of polynomials whose generating function is*

$$\sum H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^n}$$

where $A(z)$ and $B(z)$ are polynomials in z with complex coefficients. The roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on the curve C_n defined by

$$\Im \frac{B^n(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq (-1)^n \Re \frac{B^n(z)}{A(z)} \leq \frac{n^n}{(n-1)^{n-1}},$$

and are dense there as $m \rightarrow \infty$.

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